Operationalize Mathematical Sophistication in a Collaborative Problem-solving: A Conceptual Paper

Dung Tran
Australian Catholic University, Melbourne
<dung.tran@acu.edu.au>

John Munro
Australian Catholic University, Melbourne
<john.munro@acu.edu.au>

This conceptual paper proposes a model to describe the quality of student dialogue during participative collaborative problem solving. Drawing on the participation metaphor of learning, we argue that the construct of mathematical sophistication is useful to describe the quality. We then present two frameworks, mathematical competencies and mathematical practices, as ways to operationalise the construct in the collaborative problem-solving setting. We argue that by using a networking theories approach, the two frameworks will provide nuances of levels of mathematical sophistication that can be observed in student interaction. In addition, they could provide an analysis of both individual and group contributions to mathematical sophistication in a collaborative task setting. Implications of using two approaches for conceptualizing mathematical sophistication for future mathematics education research and teaching practices are provided.

Optimising the quality of students' mathematical knowledge is a major goal of mathematics education. A major focus of research has been on the knowledge gained following one or more teaching sessions. A second parallel focus that has attracted scant empirical attention is the transitory change in mathematics knowledge during learning and problem solving. The focus of this study is on these changes in knowledge as students interact with mathematics information. It proposes a conceptual framework for monitoring changes in student knowing and understanding during interactions with mathematics information in a collaborative problem-solving.

During collaborative problem-solving, students participate collaboratively to solve a problem. Analysis and description of the quality of the dialogue can inform future teacher action and researcher investigation of student learning. The shared dialogue can be recorded and unpacked in terms of its mathematical quality. In this paper, we use mathematical sophistication (Seaman & Szydlik, 2007) as a way to document the quality. Drawing on the participation metaphor of learning (Sfard, 1998), we examine why mathematical sophistication is appropriate to use for capturing the quality. We describe how the construct was initiated, and by using a networking theories approach, we operationalize it using two analytical frameworks that had been used previously for other purposes in alternative settings.

Two Metaphors of Learning, Acquisition vs. Participation, and Mathematical Sophistication

Two alternative metaphors underpin discussion about mathematical knowing and learning: the acquisition and the participation metaphors (Sfard, 1998). The acquisition metaphor focuses on the possession of mathematics knowledge, such as concepts and skills. The participation metaphor focuses on knowing and learning mathematics knowledge and skills by "doing it" in mathematical cultures. In the latter metaphor, learning is defined as legitimate peripheral participation (Lave & Wenger, 1991) or as an apprenticeship in thinking (Rogoff, 1990) in a mathematics community of practice (Bauersfeld, 1993). The teacher plays the role of a more experienced participant who inducts students into the

2019. In G. Hine, S. Blackley, & A. Cooke (Eds.). Mathematics Education Research: Impacting Practice *(Proceedings of the 42nd annual conference of the Mathematics Education Research Group of Australasia)* pp. 723-730. Perth: MERGA.

community. Students interact in it and learn its characteristic actions and language. This metaphor of learning replaces *knowledge* with *knowing*, and *having* with *doing* (Sfard, 1998).

It is possible for mathematics to be learnt through the types of actions that characterize mathematical communities (Cobb, Stephan, McClain, & Gravemeijer, 2010). Seaman and Szydlik (2007) proposed that the process by which children construct mathematical knowledge might match in some ways the practices of mathematicians when they create mathematics. They identify nine main mathematics community's values and characteristic actions. These include: (a) seeking to understanding patterns, (b) making analogies by finding the same essential structure in different mathematical objects, (c) making and testing conjectures about mathematical objects, (d) creating models and examples and non-examples of mathematical objects, (e) valuing precise mathematical definitions, (f) valuing an understanding why relationship make sense , (g) valuing logical argument and counterexamples, (h) valuing precise language and possessing fine distinctions about language, and (i) valuing symbolic representation and notation. These nine characteristic actions are referred to as mathematical sophistication traits.

Mathematical sophistication refers to the multiple avenues for knowing how to construct mathematics generally using the above characteristic actions (Seaman & Szydlik, 2007). It comprises three components: beliefs about the nature of mathematics, values concerning what it means to do mathematics, and avenues for experiencing mathematics through use of the actions. It is learnt through "enculturation into the community of practicing mathematicians" (p. 170). It applies across all content areas of mathematics rather than to "an understanding of a specific definition, mathematical object, or procedure" (p. 172).

Mathematical sophistication has been investigated both when individuals solve open-ended problems and also in multiple-choice contexts. Individuals differ in the frequency with which they use the characteristic actions and therefore their level of mathematical sophistication. Preservice teachers, for example, who ranged from average to low mathematics achievers, used them infrequently when solving mathematics problems independently (Seaman & Szydlik, 2007) and presented as "profoundly mathematically unsophisticated" with limited their problem-solving ability. Szydlik, Kuennen, and Seaman (2009) trialed a multiple-choice instrument, the Mathematical Sophistication Index (MSI) with preservice teachers. The items assessed the use of the nine sophistication traits.

It is reasonable to expect that the level of mathematical sophistication will vary between students and that what students know at a point in time about mathematical ideas and how students interact with mathematical information will influence their final understanding. Seaman and Szydlik (2007) investigated whether the possession or absence of mathematical sophistication influenced problem solving. The present conceptual paper extends this research. It examines the levels of mathematics sophistication displayed during collaborative problem solving. The motivation is mathematics educators have identified the need to teach aspects of it (e.g., National Council of Teachers of Mathematics [NCTM], 2000; Turner, Dossey, Blum, Niss, 2013).

Networking Theories Approach: The Analytical Frameworks

Mathematical sophistication is operationalised using two conceptual analytic frameworks that focus on the process aspect of learning and doing mathematics. The use of multiple interpretive theories or frameworks to study mathematics learning provides a richer understanding of it (Font Moll, Trigueros, Badillo, & Rubio 2016). The present paper conceptualises the extent to which the integrated use of multiple frameworks provides a more elaborate perspective on student learning during collaboration and a more comprehensive conceptual tool for comparing students' activity (Bikner-Ahsbahs & Prediger, 2010).

In order to operationalise the construct of mathematical sophistication during student collaborative problem solving, it is necessary to identify evidence of it in students' mathematics learning activities. The present study used two frameworks to do this: Turner et al.'s (2013) mathematical competencies and Common Core State Standards for Mathematics' (Common Core State Standards Initiative [CCSSI], 2010) set of mathematical practices.

Mathematical Competencies Framework (Turner et al., 2013)

The mathematical competencies framework (Turner et al., 2013) describes the essential actions students use when solving mathematical problems. It identified interrelated competencies: (a) dealing with mathematical rules, formalism and symbols, (b) reasoning and argumentation (logical inference), (c) solving problems mathematically (strategies), (d) communication, and (e) representation. This framework was adapted from the eight mathematical competencies specified in the KOM project (Niss & Hojgaard, 2011) that guides curriculum development, teachers training and the evaluation of student learning in Scandanavian countries. The competencies overlap with the list suggested by Seaman and Szydlik (2007), for example, reasoning and argumentation competencies match valuing logical argument and counter examples and valuing precise mathematical definition and symbolic representation and notation match the symbols and formalisms competencies. Each competency lies of a dimension of increasing complexity of mathematical thoughts and actions. Descriptors for each level of each dimension are shown in Table 1.

Table 1.
Mathematical competencies framework

Level	Symbols and formalism	Reasoning and argumentation	Solve problems mathematically	Communication	Representation
0	*No mathematical rules or symbolic expressions are used beyond basic arithmetic calculations, operating with small or easily tractable numbers*	*Make direct inferences from the instructions given.*	*Take direct actions, where the strategy needed is stated or obvious.*	*Understand a short sentence or phrase about a single familiar concept that gives immediate access to the context, where it is clear what information is relevant, and where the order of information matches the required steps of thought.*	*Directly handle a given representation where minimal interpretation is required for example, go from text to numbers, read a value directly from a graph or table.*
1	*Make direct use of a simple functional relationship, either implicit or explicit; use formal*	*Infer by linking information, (for example, link separate components in a problem, or use direct*	*Decide on a suitable strategy that uses the relevant given information to*	*Identify and extract relevant information. Use links within a text to understand the context and task, or between the*	*Select and interpret one standard or familiar representation in relation to a situation*

	mathematical symbols or use directly a formal mathematical definition, convention or symbolic concept	*reasoning within one aspect of the problem).*	*reach a conclusion*	*text and other related representations. Constructive communication is simple, but is more than the presentation of a single numeric result*	
2	*Use and manipulate symbols explicitly; use mathematical rules, definitions, convent-ions, procedures or formulae using a combination of multiple relationships or symbolic concepts.*	*Analyse information (for example to connect several variables) to follow or create a multi-step argument; reason from linked information sources.*	*Construct a strategy to transform given information to reach a conclusion.*	*Revisit the text to understand instructions and decode the elements of the context or task; interpret conditional statements or instructions containing multiple elements; or actively communicate a constructed description or explanation.*	*Translate between or use two or more different representation s in relation to a situation, including modifying a representation; or devise a simple representation of a situation.*
3	*Multi-step application of formal mathematical procedures; working flexibly with functional or involved algebraic relationships; using both mathematical technique and knowledge to produce results.*	*Synthesize and evaluate, use or create chains of reasoning to justify inferences or to make generalization s, drawing on and combining multiple elements of information in a sustained and directed way*	*Construct an elaborated strategy to find an exhaustive solution or a generalized conclusion; evaluate or compare strategies.*	*Create an economical, clear, coherent and complete description or explanation of a solution, process or argument; interpret complex logical relations involving multiple ideas and connections.*	*Understand and use a non-standard representation that requires substantial decoding and interpretation; or devise a representation that captures the key aspects of a complex situation; or compare or evaluate representation.*

The competencies were used to describe individual students' levels of proficiency in PISA mathematics assessments and predicted 70% of the variability in the difficulty of PISA tasks (Turner et al., 2013).

The mathematical competencies framework, by representing increasing complexity of mathematical thoughts and actions, can provide a lens to capture the nuances of sophistication students exhibit when solving mathematical problems. We propose that this framework can be used in a different way, to examine the interactions between students during collaborative problem solving and to infer their level of sophistication. However, the framework does not assist in describing the quality of individual students' contributions; the *mathematical practices framework* is used for this.

Mathematical Practices Framework (CCSSI, 2010)

Some researchers (e.g., Cobb et al., 2010; Selling, 2016) have described mathematicians' ways of thinking and doing mathematics as "mathematical practices" (MPs). A variety of MPs have been identified (e.g., CCSSI, 2010; NCTM 2000; RAND Mathematics Study Panel, 2003). They are linked with the development of conceptual understanding (Boaler & Staples, 2008) and positive dispositions of learners (Cobb, Gresalfi, & Hodge, 2009).

MPs in a classroom can be conceptualized as both emergent, following normative ways of working and disciplinary (mathematics) ways of working (Selling, 2016). Earlier studies examined the MPs in specific mathematics topics (e.g., Cobb et al., 2010). The present study examines their use more generally, across multiple topics in mathematics (CCSSI, 2010; Selling, 2016).

The use of individual actions during mathematics learning has been investigated extensively. This includes representing (e.g., Goldin, 1998), generalizing (e.g., Carraher, Martinez, & Schliemann, 2008), problem solving (e.g., Hiebert et al., 1996; Schoenfeld, 1992), and justifying (e.g., Ball & Bass, 2003). What has received less attention is the use of two or more actions during a mathematics learning event. The next section describes studies that examine this.

The USA curriculum standards document (CCSSI, 2010) includes eight MPs: making sense of problems and persevering in solving them (MP1), reasoning abstractly and quantitatively (MP2), constructing viable arguments and critiquing the reasoning of others (MP3), modelling with mathematics (MP4), using appropriate tools strategically (MP5), attending to precision (MP6), looking for and making use of structure (MP7), and looking for and expressing regularity in repeated reasoning (MP8). This list is similar to that of Seaman and Szydlik (2007) in identifying verbs that describe how mathematicians work. It also specifies mathematical behaviours that can be promoted in classrooms. It can be used to analyse student mathematical sophistication in a collaborative problem-solving setting in which there is limited teacher intervention.

Teachers can scaffold and direct students to use the MPs through explicit teaching (Selling, 2016). However, a goal of mathematics education is that students learn to use them independently and to self-direct their use. We propose that teachers and researchers can observe independent student use of the actions, that is, without explicit scaffolding by an expert participant. In addition, by identifying the student who initiates an MP and how the MP is maintained, developed or ignored in the group, teachers and researchers can unpack individual contributions in the collaborative setting.

Why Do We Need Two Frameworks?

In collaborative problem-solving contexts, two or more students interact to develop a solution to a problem that can be solved in multiple ways. When investigating the thinking processes the students use, researchers analyse what they know and how they think. Their problem-solving activity is inferred from the sequentially organised statements they share. This dialogue can be analysed for evidence both of mathematical competencies and the use of MPs. The quality of their mathematics actions is described by their competencies (Turner et al., 2013). How they think and use their actions is captured by the mathematical practices (CCSSI, 2010). The two frameworks can be used in a networked or synthesised way. The actions repertoire the students use can be described as interactions or networking between the two frameworks.

The two frameworks can be used to infer mathematical sophistication in classroom settings more generally. They provide complementary lenses both on interactions between the students in a collaborative group and on individual student thinking. The competencies framework is developmental or hierarchical in its nature while the MPs framework describes the breadth or range of mathematical behaviours used at each level of the hierarchy. Together they potentially capture the nuances of mathematical sophistication in students' reasoning by focusing on the *experience* aspect (Seamna & Szydlik, 2007).

This experience aspect in collaborative problem-solving settings includes both the process aspect of doing mathematics during the collaboration and the transitory knowledge contributions of individuals during this activity. An observer can track changes in the dyads joint/negotiated understanding of the task and the thinking participants used to advance their understanding. Comparisons across dyads/groups permit the identification of more sophisticated mathematical competence and the use of the MPs.

Summary

The conceptual framework described in this paper provides a lens for monitoring, analysing and evaluating students' interactions during collaborative problem solving. The analysis and evaluation using the networking theories approach offer an insight into individual student thinking at any time and the quality of their interaction. Together these provide an insight into mathematical sophistication and its role in problem solving and mathematics learning more generally. It extends Tran and Chan's (2017) work by elaborating an operational definition of sophistication and providing greater insight into the nuances of individual students' contributions in a collaborative problem-solving setting.

The metaphor of learning as participation underpins the study, with its focus on the quality of a student's contribution to the knowledge of the group at any point in time and the process associated with it. The conceptual framework draws on the student's transitory mathematics knowledge at any time. This is the focus on mathematics knowing rather than on fixed mathematical knowledge. It is the trajectory through the various states of knowing during a period of learning that leads to mathematics knowledge. Any stage of knowing on the trajectory is influenced by the student's level of mathematical sophistication as a mathematical thinker.

An implication of this conceptual framework is that mathematics education might benefit from the analysis of mathematical sophistication during problem solving and learning generally. To this end, future research could investigate empirically the framework and offer more nuanced versions to describe a range of levels of mathematical sophistication.

Implications for Teaching and Pedagogy

This conceptual paper draws attention to the value of focusing on transitory knowing and the negotiation of meaning during mathematics learning and the need to do this. The framework developed in the paper opens up this perspective on mathematics learning and its implications for enhancing mathematics knowledge. The mathematical competencies framework provides means to identify levels of sophistication of students' way of knowing. Teachers could use these data formatively to inform their pedagogical actions in order to advance their competencies towards the characteristic behaviours of mathematicians. The breadth of MPs used by students also has an impact on mathematical sophistication. To broaden the range of MPs used by students, teachers can provide a range of learning opportunities that elicit the appropriate range of MPs.

Acknowledgment

The writing is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 503.01-2015.02.

References

Bell, A. W. (1979). The learning of process aspects of mathematics. *Educational Studies in Mathematics*, *10*(3), 361-387.

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—an approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers. Advances in mathematics education series* (Vol. 1, pp. 483–506). New York: Springer.

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 27–44). Reston, VA: National Council of Teachers of Mathematics.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record*, *110*(3), 608-645.

Bauersfeld, H. (1993, March). Teachers pre and in-service education for mathematics teaching. *Seminaire sur la Representation*, No. 78, CIRADE, Universite du Quebec ' Montreal.

Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM*, *40*(1), 3–22. doi:10.1007/s11858-007-0067-7

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education*, *40*(1), 40-68.

Cobb, P., Stephan, M., McClain, K. & Gravemeijer, K. (2010). Participating in classroom mathematical practices. In E. Yackel, K. Gravemeijer, & A. Sfard (Eds). *A journey in mathematics education research – Insights from the work of Paul Cobb* (pp. 117-163). Netherlands: Springer.

Common Core State Standards Initiative [CCSSI]. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Font Moll, V., Trigueros, M., Badillo, E., & Rubio, N. (2016). Mathematical objects through the lens of two different theoretical perspectives: APOS and OSA. *Educational Studies in Mathematics*, *91*(1), 107-122. doi:10.1007/s10649-015-9639-6

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behavior*, *17*(2), 137–165. doi:10.1016/S0364-0213(99)80056-1

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., . . . & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher*, *25*(4), 12–21. doi:10.3102/0013189X025004012

National Council of Teachers of Mathematics [NCTM] (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Niss, M., & Højgaard, T. (2011). *Competencies and mathematical learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark* (English edition). pure. au. dk/portal/files/41669781/THJ11_MN_KOM_in_english. pdf.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND. Retrieved from http://www.rand.org/pubs/monograph_reports/MR1643.html

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan.

Seaman, C. E., & Szydlik, J. E. (2007). Mathematical sophistication among preservice elementary teachers. *Journal of Mathematics Teacher Education*, *10*(3), 167-182. doi:10.1007/s10857-007-9033-0

Selling, S. K. (2016). Making mathematical practices explicit in urban middle and high school mathematics classrooms. *Journal for Research in Mathematics Education*, *47*(5), 505-551.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher*, *27*(2), 4-13.

Szydlik, J., Kuennen, E., Seaman C. (2009). Development of an Instrument to Measure Mathematical Sophistication. *Proceedings for the Twelfth Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education Conference on Research in Undergraduate Mathematics Education*.

Tran, D., & Chan, M. C. E. (2017). Examining mathematical sophistications in collaborative problem-solving. In B. Kaur, W. K. Ho, T. L. Toh & B. H. Choy (Eds.), *Proceedings of the 41st Annual Meeting of the International Group for the Psychology of Mathematics Education*, Singapore, Vol. 4 (pp. 281-288).

Turner, R., Dossey, J., Blum, W., & Niss, M. (2013). Using mathematical competencies to predict item difficulty in PISA: A MEG study. In M. Prenzel, M. Kobarg, K. Schops, & S. Ronnebeck (Eds*). Research on PISA – Research outcomes of the PISA Research Conference 2009* (pp. 23-37). Netherlands: Springer.